Project On Multi-A(ge)nt Systems (236824)

1 Introduction and overview

The paper [1] uses the tools of algebraic graph theory to develop necessary and sufficient conditions for consensus of multi-agent system with second-order dynamics by measuring both the current and the sampled past state of the position (hybrid system), without the use of velocity measurement or without the need to store the whole spectrum of the delayed position information for more the one sampling time at a time. In addition, this paper presents results that can be applied to undirected and directed graphs alike.

One extremely interesting behavior is consensus, which refers to reaching an agreement among a group of autonomous agents. Most of the studies dealing with the consensus protocol refer to the agents as a collection of agents having first-order dynamics [2]. Recently, a more realistic analysis of the multi-agent dynamical systems models the agents as a collection of agents having second-order dynamics [3].

When dealing with realistic dynamical systems, some ideal assumptions must be relaxed. The studied cases in the literature include one way communication (Directed Graphs) [2], time delays [3], sampling time and output feedback (in our paper the unavailability of velocity states) [1]. It was found that both the real and imaginary parts of the eigenvalues of the Laplacian matrix associated with the corresponding network topology play key roles in reaching second-consensus. It was shown [2][4] that by using information of both the current and delayed position states, the sampled position of the multi-agent system converges faster than the standard consensus multi-agent system, or even that the consensus of a multi-agent system cannot be reached [3][5] without delayed position information under the given protocol but it can be achieved with an even relatively small time delay by appropriately choosing the coupling strengths.

The paper shows that by choosing the appropriate sampling period, a consensus of multi-agent system can be reached if and only if the sampling period is chosen from some particular time intervals depending on the coupling strengths and the spectrum of the Laplacian matrix of the network.

2 Mathematical concepts

In the current chapter we will present a discussion on the mathematical tools used in this paper, including a description of relevant notations and definitions.

Basic concepts of algebraic graph theory:

- The set of nodes $\mathrm{V} = \{\mathrm{v}_1,\mathrm{v}_2,..\ \mathrm{v}_\mathrm{N}\}$

- The set of directed edges $E \subseteq V \times V$
- The weighted adjacency matrix $G = (G_{ij})_{N\times N}$
- A directed edge e_{ij} in network G is denoted by the ordered pair of nodes (v_i,v_j), where v_i and v_j are called the parent and child nodes, respectively (meaning that node v_j can receive information from v_i
- A directed path from node v_i to node v_j in G is a sequence of edges (v_i,v_{i1}), (v_{i1},v_{i2}), . . . , (v_{il},v_j) in the directed network with distinct nodes v_{ik} , k = 1, 2, . . . , l
- A root r is a node such that for each node v different from r , there is a directed path from r to v
- A directed tree is a directed graph, in which there is exactly one root and every node except for this root itself has exactly one parent
- A directed spanning tree is a directed tree consisting of all the nodes and some edges in G
- A directed graph contains a directed spanning tree if one of its subgraphs is a directed spanning tree
- The Laplacian matrix $L = (L_{ij})_{N\times N}$ is defined by:

$$L_{ii} = -\sum_{j=1, j\neq i}^{N} L_{ij} \quad , \; L_{ij} = -G_{ij} \quad , j \neq i \tag{1}$$

which ensures the diffusion property that $\sum_{j=1,}^{N} L_{ij} = 0$.

For notational simplicity, n = 1 is considered throughout the paper, but all the results obtained can be easily generalized to the case with n > 1 by using the Kronecker product operations.

Now, let us connect these definitions to multi agent systems:

The second-order consensus protocol in multi-agent dynamical systems can be represented by:

$$\begin{gathered} \dot{x}_i(t) = v_i \\ \dot{v}_i(t) = \tilde{\alpha} \sum_{j=1, j\neq i}^{N} G_{ij} \left(x_j(t) - x_i(t)\right) + \tilde{\beta} \sum_{j=1, j\neq i}^{N} G_{ij} \left(v_j(t) - v_i(t)\right) \; , \{i = 1, 2, \ldots N\} \end{gathered} \tag{2}$$

where $x_i \in R^n$ and $v_i \in R^n$ are the position and velocity states of the i-th agent (node), respectively, $\widetilde{\alpha} > 0$ and $\widetilde{\beta} > 0$ are the coupling strengths, and $G = (G_{ij})_{N\times N}$ is the coupling configuration matrix representing the topological structure of the network and thus is the weighted adjacency matrix of the network.

The paper will show that a second order consensus can be reached under some conditions, even if the velocity states are unavailable. To do so, the following consensus protocol with both current and sampled position data is considered (using Laplacian notation):

$$\begin{gathered} \dot{x}_i(t) = v_i \\ \dot{v}_i(t) = \alpha \sum_{j=1}^{N} G_{ij}(x_j(t) - x_i(t)) - \beta \sum_{j=1}^{N} G_{ij}(x_j(t_k) - x_i(t_k)) \\ = -\alpha \sum_{j=1}^{N} L_{ij} x_j(t) + \beta \sum_{j=1}^{N} L_{ij} x_j(t_k) \quad, t \in [t_k, t_{k+1}) \ , i = 1, 2, \ldots N \end{gathered} \tag{3}$$

where t_k are the sampling instants satisfying $0 = t_0 < t_1 < \cdots < t_k < \cdots$, and α and β are the coupling strengths. For simplicity, assume that $t_k + 1 - tk = T$, where $T > 0$ is the sampling period.

Definition 1. The consensus of the multi-agent system is defined as:

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \quad \lim_{t\to\infty} \|v_i(t) - v_j(t)\| = 0 \quad \forall i, \quad j = 1, 2, \ldots N \tag{4}$$

Let us define a few more useful definitions to fully define our mathematical tool set:

- Let R(u) and I(u) be the real and imaginary parts of a complex number u, $0 = \mu_1 \leq R(\mu_2) \leq \cdots \leq R(\mu_N)$ be the N eigenvalues of the Laplacian matrix L, $I_m \in R^{m\times m}$ ($O_N \in R^{m\times m}$) be the m-dimensional identity (zero) matrix, $1_m \in R^m$ ($0_N \in R^m$) be the vector with all entries being 1 (0), and $\|a_1 + ia_2\| = \sqrt{a_1^2 + a_2^2}$ be the norm of a complex number $a_1 + ia_2$ where $i = \sqrt{-1}$.

- **Lemma 1.** The Laplacian matrix L has a simple eigenvalue 0 and all the other eigenvalues have positive real parts if and only if the directed network has a directed spanning tree.

- **Lemma 2.** The Kronecker product $\otimes$ has the following properties: For matrices A, B, C and D of appropriate dimensions:

  - $(A + B) \otimes C = A \otimes C + B \otimes C$
  - $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

- **Lemma 3.** Given a complex coefficient polynomial of order two as follows:

$$g\left(s\right)=s^{2}+\left(\varrho_{1}+i\gamma_{1}\right)s+\varrho_{0}+i\gamma_{0} \tag{5}$$

where $\varrho_{1},\ \gamma_{1},\ \varrho_{0},\ \gamma_{0}$ are real constants. Then, g (s) is stable if and only if $\varrho_{1}>0$ and $\varrho_{1}\gamma_{1}\gamma_{0}+\varrho_{1}^{2}\varrho_{0}-\gamma_{0}^{2}>0$.

3 Main Paper Contribution

This chapter presents a detailed discussion of the main results of the paper. Several proofs of theorems and corollaries are not presented in the chapter due to space limitation and the reader is kindly referred to the paper for detailed proofs.

Let $\eta_{i}=\left(x_{i},v_{i}\right)^{T},A=\begin{pmatrix}0&1\\0&0\end{pmatrix},and$ B$=\begin{pmatrix}0&0\\1&0\end{pmatrix}$. Then the system Eq. (3) can be rewritten as

$$\begin{aligned}&\dot{\eta}_{i}\left(t\right)=A\eta_{i}-\alpha\sum_{j=1}^{N}L_{ij}B\eta_{j}\left(t\right)\\&+\beta\sum_{j=1}^{N}L_{ij}\eta_{j}\left(t_{k}\right)\quad,\quad t\in\left[t_{k},t_{k+1}\right)\ ,i=1,2,\ldots N\end{aligned} \tag{6}$$

And the solution of an isolate node system of Eq. (6) satisfies:

$$\dot{s}_{i}\left(t\right)=As\left(t\right),\quad t\in\left[t_{k},t_{k+1}\right) \tag{7}$$

where $s\left(t\right)=\left(s_{1},s_{2}\right)^{T}$ is the state vector. Let $\eta=\left(\eta_{1}^{T},\ldots\eta_{N}^{T}\right)^{T}$, Eq. (7) becomes:

$$\dot{\eta}\left(t\right)=\left[\left(I_{N}\otimes A\right)-\ \alpha\left(L\otimes B\right)\right]\eta\left(t\right)+\beta\left(L\otimes B\right)\eta\left(t_{k}\right),\quad t\in\left[t_{k},t_{k+1}\right) \tag{8}$$

Let J be the Jordan form associated with the Laplacian matrix L, i.e., L $=$ PJP^{-1} , where P is a nonsingular matrix, let $y\left(t\right)=\left(\left(P^{-1}\otimes I_{2}\right)\eta\left(t\right)\right.$. By Lemma 2, one has:

$$\dot{y}\left(t\right)=\left[\left(I_{N}\otimes A\right)-\ \alpha\ \left(J\otimes B\right)\right]y\left(t\right)+\beta\left(J\otimes B\right)y\left(t_{k}\right),\quad t\in\left[t_{k},t_{k+1}\right) \tag{9}$$

If the graph G is undirected, then L is symmetric and J is a diagonal matrix with real eigenvalues. However, when G is directed, some eigenvalues of L may be complex, and J = diag(J_1, J_2, . . . , J_r), where

$$J_l = \begin{pmatrix} \mu_l & 0 & 0 & 0 \\ 1 & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & \mu_l \end{pmatrix}_{N_l \times N_l} \tag{10}$$

in which μ_l are the eigenvalues of the Laplacian matrix L, with multiplicity N_l , l = 1, 2, . . . , r , $N_1 + N_2 + \cdots + N_r = N$. Note that if the network G contains a directed spanning tree, then by Lemma 1, 0 is a simple eigenvalue of the Laplacian matrix L, so:

$$\dot{y}_1(t) = Ay_1(t), \quad t \in [t_k, t_{k+1}) \tag{11}$$

Theorem 1. Suppose that the network G contains a directed spanning tree. Then, second-order consensus in system Eq. (3) can be reached if and only if, in Eq. (9),

$$\lim_{t \to \infty} \|y_i\| \to 0, \quad i = 2, \ldots N \tag{12}$$

Corollary 1. Suppose that the network G contains a directed spanning tree. Then, second-order consensus in system Eq. (3) can be reached if and only if the following N − 1 systems are asymptotically stable:

$$\dot{z}_i(t) = (A - \alpha\mu_i B) z_i(t) + \beta\mu_i B z_i(t_k), \quad t \in [t_k, t_{k+1}) \ , i = 2, \ldots N \tag{13}$$

Until now, it is still very hard to check the conditions Eq. (12) and Eq. (13) in Theorem 1 and Corollary 1 which do not reveal how network structure affects the consensus behavior. Next, a theorem is derived to ensure consensus depending on the control gains, spectra of the Laplacian matrix, and the sampling period.

Theorem 2. Suppose that the network G contains a directed spanning tree. Then, second-order consensus in system Eq. (3) can be reached if and only if

$$0 < \frac{\beta}{\alpha} < 1 \tag{14}$$

and

$$f\left(\alpha,\beta,\mu_i,T\right)=\frac{\left(\frac{\beta}{\alpha}\right)^2}{1-\left(\frac{\beta}{\alpha}\right)}\left(\sin^2\left(d_iT\right)-\sinh^2\left(c_iT\right)\right)\times\left(\cosh\left(c_iT\right)-cos\left(d_iT\right)\right)^2 \\ -4\sin^2\left(d_iT\right)\sinh^2\left(c_iT\right)>0 \quad ,i=2,\ldots N \tag{15}$$

where $c_i=\sqrt{\frac{|\alpha|(\|\mu_i\|-sign(\alpha)R(\mu_i))}{2}}$ and $d_i=\sqrt{\frac{|\alpha|(\|\mu_i\|+sign(\alpha)R(\mu_i))}{2}}$

The proof uses mainly **Corollary 1** and **Lemma 3.**

Remark 1. In Theorem 2, a necessary and sufficient condition for second-order consensus in the multi-agent dynamical system Eq. (3) is established. For a given network, one can design appropriate α , β , and T such that the conditions Eq. (14) and Eq. (15) in Theorem 2 are satisfied. It is interesting to see that f increases as the parameter β/α increases. Thus, one can choose a large value of β/α such that Eq. (15) holds. Since the condition Eq. (15) holds for all i =2, . . . , N, one can find a stable consensus region as follows: S $=\{$ c +id | f (α , β , c + id, T) $>$ $0\}$, where c and d are real. Then, the problem is transformed to finding if all the nonzero eigenvalues of the Laplacian matrix lie in the stable consensus region S, i.e., $\mu_i \in$ S for all i $=$ 2, . . . , N. In this paper, by introducing sampled position data in the consensus algorithm, it will be shown in the simulation that there exist some disconnected regions for choosing appropriate sampling periods.

Corollary 2. Suppose that the network G contains a directed spanning tree and all the eigenvalues of its Laplacian matrix are real. Then, second-order consensus in system Eq. (3) can be reached if and only if

$$0<\beta<\alpha \tag{16}$$

and

$$\sqrt{\alpha\mu_i}T\neq k\pi \quad , \quad i=2,\ldots N,k=0,1,\ldots \tag{17}$$

Remark 2. If all the eigenvalues of the Laplacian matrix are real, which includes the undirected network as a special case, then the condition Eq. (7), i.e., $T\neq\frac{k\pi}{\sqrt{\alpha\mu_i}}$, is very easy to be verified and applied. It is quite interesting to see that second-order consensus in the multi-agent system Eq. (3) can be reached if and only if $0<\beta<\alpha$ and the sampling period T is not of some particular value.

Usually, the convergence rate around the critical points $T=\frac{k\pi}{\sqrt{\alpha\mu_i}}$ is very slow. Therefore, it is hard to achieve better performance for a large T in a very large-scale network. A corollary is given below to simplify the theoretical analysis.

Corollary 3. Suppose that the network G contains a directed spanning tree and all the eigenvalues of its Laplacian matrix are real. Then, second-order consensus in system Eq. (3) can be reached if Eq. (16) is satisfied and

$$0 < T < \frac{\pi}{\sqrt{\alpha \mu_N}} \tag{18}$$

Corollary 3 implies that if the network G contains a directed spanning tree and all the eigenvalues of the Laplacian matrix are real, then second-order consensus in system Eq. (3) can be reached provided that the sampling period is less than the critical value $\frac{\pi}{\sqrt{\alpha \mu_N}}$ depending on the largest eigenvalue of the Laplacian matrix. However, to our (mine and the paper's authors) surprise, second-order consensus in system Eq. (3) cannot be reached in a general directed network with complex Laplacian eigenvalues for a sufficiently small or a sufficiently large sampling period T .

Corollary 4. Suppose that the network G contains a directed spanning tree and there is at least one eigenvalue of its Laplacian matrix with a nonzero imaginary part. Then, second-order consensus in the system Eq. (3) cannot be reached for a sufficiently small or a sufficiently large sampling period T.

Remark 3. If the network G contains a directed spanning tree and all the eigenvalues of the Laplacian matrix are real, then second-order consensus in system Eq. (3) can be reached for a sufficiently small sampling period T as stated in Corollary 2. However, if there is at least one eigenvalue of the Laplacian matrix having a nonzero imaginary part, then second-order consensus cannot be reached for a sufficiently small sampling period T as shown in Corollary 4, which is inconsistent with the common intuition that the consensus protocol Eq. (3) should be better if the sampled information is more accurate for a small sampling period. Interestingly, the nonzero imaginary part of the eigenvalue of the Laplacian matrix leads to possible instability of consensus.

4 Results

The paper presents an undirected topology example, for which the authors present numerical results. In this chapter we will reproduce those results. The topology is given in Table 1.

L	μ_1	μ_2	μ_3	μ_4	$\frac{k\pi}{\sqrt{\alpha\mu_2}}$	$\frac{k\pi}{\sqrt{\alpha\mu_3}}$	$\frac{k\pi}{\sqrt{\alpha\mu_4}}$
$\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$	0	1	3	4	3.1416	1.8138	1.5708

Table 1: Parameters.

The topology was analyzed for three cases, described in Table 2.

	Sample Time	α	β	Position and velocity	Is there consensus?
Case 1	T=1[s]	1	0.8	Figure 1	Yes
Case 2	T=0.1[s]	1	0.8	Figure 2	Yes
Case 3	T= 1[s]	1	1.1	Figure 3	No

Table 2: Simulation parameters and consensus results in NetLogo.

The position and velocity states of the agents for the cases 1,2,3 are given in Figure 1, Figure 2, Figure 3 respectively.

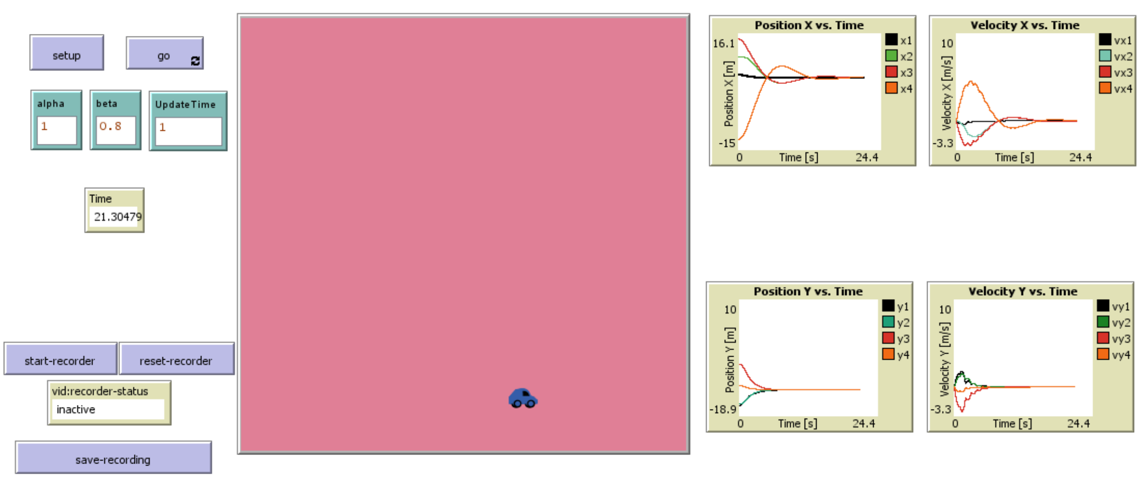

Figure 1: Position and velocity states of the agents, Case 1.

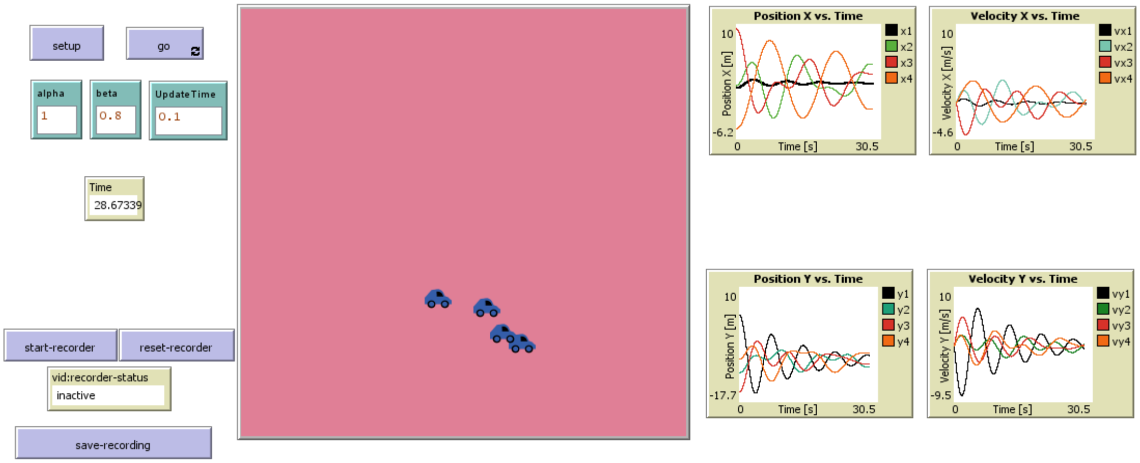

Figure 2: Position and velocity states of the agents, Case 2.

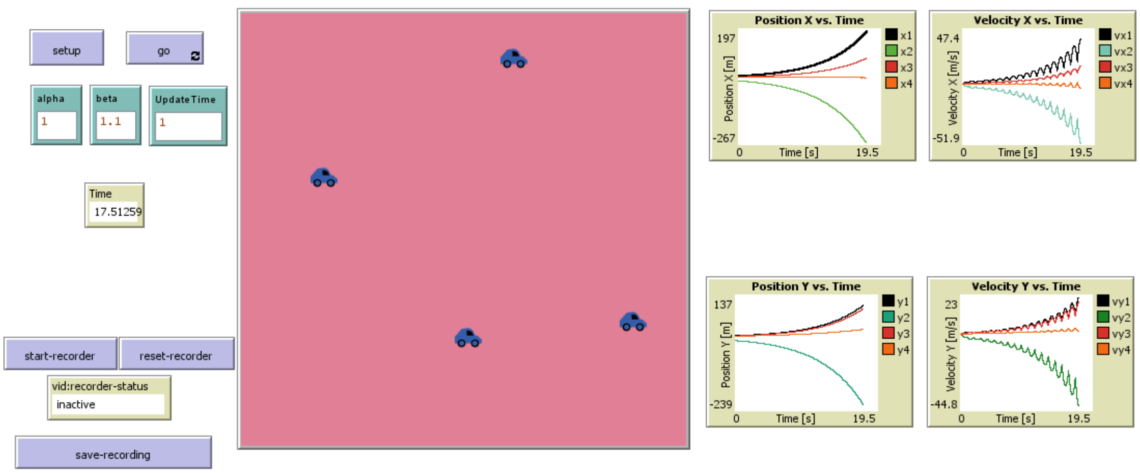

Figure 3: Position and velocity states of the agents, Case 3.

5 Conclusions

The following chapter summarizes the pros and cons of the paper.

Summary: It was shown that if all the eigenvalues of the Laplacian matrix are real, then second order consensus in the multi-agent system can be reached for any sampling period except at some critical points depending on the spectrum of the Laplacian matrix. However, if there exists at least one eigenvalue of the Laplacian matrix with a nonzero imaginary part, second-order consensus cannot be reached for sufficiently small or sufficiently large sampling periods. In such cases, one nevertheless may be able to find some disconnected stable consensus regions determined by choosing appropriate sampling periods. As to future work, the authors presented several new branches of research that they would like to follow: study of multi-agent systems with non-uniform sampling intervals, nonlinear dynamics with time-varying velocities, more general consensus protocols.

A necessary and sufficient condition for reaching consensus in multi-agent dynamical systems with a general directed network topology is established and demonstrated. By using only sampled position data in this paper and without requiring the velocity information of agents in second-order dynamics, it is found in this paper that second-order consensus in multi-agent system can be reached by appropriately choosing the sampling period. Nevertheless, no detailed analysis of the rate of convergence was given. In addition, only the graphs with directed spanning tree were analyzed, without addressing the graphs without a directed spanning tree.

References

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